Reduced Order Model Approach to Inverse Scattering

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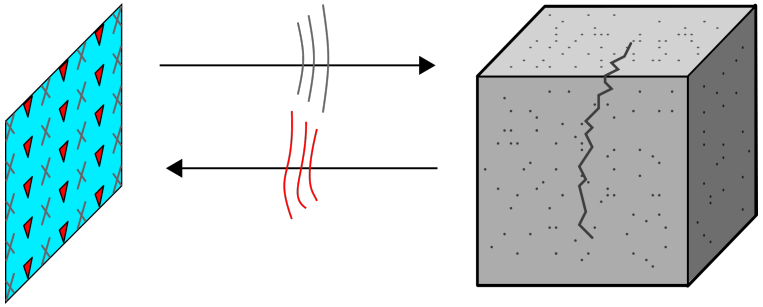
22 April 2020, ICERM & Jörn's Couch

Introduction

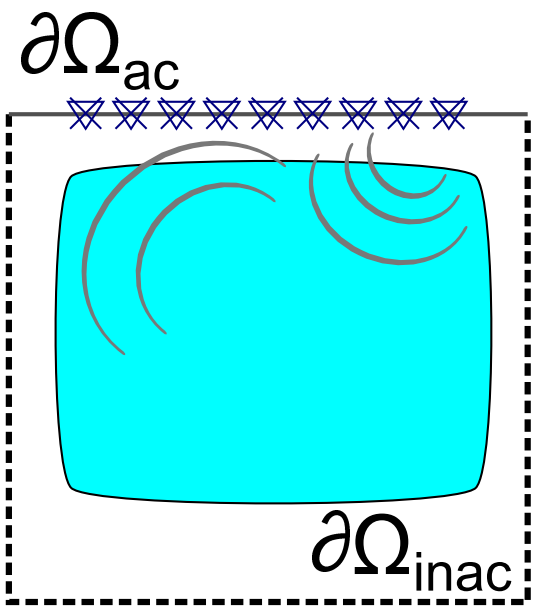


Consider inverse scattering for a hyperbolic equation

$$(\partial_t^2 + L_q L_q^T)\mathbf{u}(t,\mathbf{x}) = 0$$
$$\mathbf{u}(0,\mathbf{x}) = \mathbf{b}(\mathbf{x}) \qquad \partial_t \mathbf{u}(0,\mathbf{x}) = 0$$

with first order waveop. L_q affine in reflectivity q as

$$L_q^T \mathbf{u} = \sqrt{c}\nabla[\sqrt{c}\mathbf{u}] + \frac{c}{2}\nabla q\mathbf{u}$$



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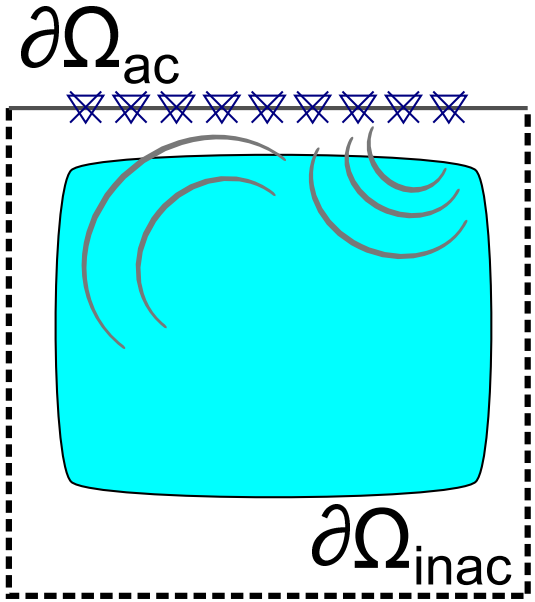

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$$L_q^T \mathbf{u} = \sqrt{c}\nabla[\sqrt{c}\mathbf{u}] + \frac{c}{2}\nabla q \mathbf{u}$$

Coinciding sources and receivers yield data at times $t = j\tau$ for $j = 0, 1, \ldots, 2n-1$.

$$\mathbf{D}_j = \langle \mathbf{b}(x), \mathbf{u}(j\tau, x)\rangle = \left\langle \mathbf{b}(x), \cos\left(j\tau\sqrt{L_q L_q^T}\right)\mathbf{b}(x)\right\rangle$$

In a MIMO setting $\mathbf{b}(x) = (b^{(1)}(x), ..., b^{(m)}(x))$ with m sources/receivers

Problem

1. Find reflectivity $q(x)$ from backscattering data $\mathbf{D}$, i.e. invert $q \mapsto \mathbf{D}$

Outline of approach

1. Construct a ROM $\mathcal{L}_q^{ROM}$ from the measured data that has the structure of a wave operator
   - Data interpolation

$$\mathbf{D} = \left\langle \mathbf{b}(x), \cos\left(j\tau\sqrt{L_q L_q^T}\right)\mathbf{b}(x)\right\rangle$$
$$= \left\langle \mathbf{b}^{ROM}, \cos\left(j\tau\sqrt{\mathcal{L}_q^{ROM}\mathcal{L}_q^{ROM\,T}}\right)\mathbf{b}^{ROM}\right\rangle$$

   - Sparsity pattern of $\mathcal{L}_q^{ROM}$ should resemble discretization
   - Regularization on level of the ROM

2. Interpret ROM $\mathcal{L}_q^{ROM}$ as a wave operator (affine in q)
3. Invert for the reflectivity by minimizing ROM mismatch

$$\mathcal{O}^{LS}(q^s) = ||\mathcal{L}_q^{ROM} - \mathcal{L}_{q^s}^{ROM}|| + \textit{regularization}$$

Comparison to standard imaging (FWI)

FWI

1. Objective function data mismatch
2. Highly nonlinear (many iterations)
3. Difficult to regularize

ROM Approach

1. Objective function ROM missmatch
2. Close to linear due to ROM transform
3. Intrinsic regularization

The Propagator

- We define the Propagator $\mathcal{P}_q = \cos\left(\tau\sqrt{L_q L_q^T}\right)$
- Field can be expressed in Chebyshev polynomials of first kind of $\mathcal{P}_q$

$$\begin{aligned}\mathbf{u}(j\tau, x) &= \cos\left(j\tau\sqrt{L_q L_q^T}\right)\mathbf{b} = \cos\left(j\arccos\left[\cos\left(\tau\sqrt{L_q L_q^T}\right)\right]\right)\mathbf{b} \\ &= \cos(j\arccos[\mathcal{P}_q])\,\mathbf{b} = \mathcal{T}_j(\mathcal{P}_q)\mathbf{b}\end{aligned}$$

- We move from continuous time

$$\partial_t^2\mathbf{u}(t,\mathbf{x}) = -L_q L_q^T\mathbf{u}(t,\mathbf{x}) \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{b}(\mathbf{x}) \quad \partial_t\mathbf{u}(0,\mathbf{x}) = 0 \tag{1}$$

to a discrete time equation $\mathbf{u}_j = \mathbf{u}(j\tau, x)$ (Chebyshev recursion)

$$\mathbf{u}_{j+1} = 2\mathcal{P}_q\mathbf{u}_j - \mathbf{u}_{j-1} \quad \mathbf{u}_0 = \mathbf{b}(x) \quad \mathbf{u}_1 = \mathbf{u}_{-1} \tag{2}$$

- Note the similarity of (2) to discretizing ∂_t^2 in (1)

$$\frac{\mathbf{u}_{j+1} - 2\mathbf{u}_j + \mathbf{u}_{j-1}}{\tau^2} = \frac{2}{\tau^2}(\mathcal{P}_q - \mathcal{I})\mathbf{u}_j = -\mathcal{L}_q\mathcal{L}_q^T\mathbf{u}_j \tag{3}$$

Galerkin projection - ROM

- Project (2) onto the solutions at the first n times

$$\mathbf{U}(x) = (\mathbf{u}_0(x), \mathbf{u}_1(x), \ldots, \mathbf{u}_{n-1}(x))$$

- Expand $\mathbf{u}_j \approx \mathbf{U}(x)\mathrm{g}_j$ and use the Galerkin condition to find

$$\langle \mathbf{U}, \mathbf{U} \rangle \mathrm{g}_{j+1} = 2\langle \mathbf{U}, \mathcal{P}_q \mathbf{U} \rangle \mathrm{g}_j - \langle \mathbf{U}, \mathbf{U} \rangle \mathrm{g}_{j-1}$$

$$\mathrm{g}_0 = \mathrm{e}_1 \qquad \mathrm{g}_1 = \mathrm{g}_{-1} \quad \mathbf{D}_j^{ROM} = \mathrm{g}_0^T \langle \mathbf{U}, \mathbf{U} \rangle \mathrm{g}_j$$

- This defines our ROM with

$$\mathrm{M}\mathrm{g}_{j+1} = 2\mathrm{S}\mathrm{g}_j - \mathrm{M}\mathrm{g}_{j-1}$$

Propositions

1. The data obtained from the ROM interpolates the true data

$$\mathbf{D}_j^{ROM} = \mathbf{D}_j \quad j \leq 2n - 1$$

2. We can find M and S from the data $\mathbf{D}_j$

Reduced order model propagator operator I

$$\mathrm{M}\mathrm{g}_{j+1} = 2\mathrm{S}\mathrm{g}_j - \mathrm{M}\mathrm{g}_{j-1}$$

- Cholseky factorize $\mathrm{M} = \mathrm{R}^T\mathrm{R}$ to orthogonalize the basis $\mathbf{V} = \mathbf{U}\mathrm{R}^{-1}$
- Introduce ROM field $\mathrm{u}_j^{ROM} = \mathrm{R}\mathrm{g}_j = \left\langle \mathbf{U}\mathrm{R}^{-1}, \mathbf{u}_j \right\rangle$

$$\mathrm{u}_{j+1}^{ROM} = 2\mathrm{R}^{-T}\mathrm{S}\mathrm{R}^{-1}\mathrm{u}_j^{ROM} - \mathrm{u}_{j-1}^{ROM} \qquad \mathrm{u}_0^{ROM} = \mathrm{R}\mathrm{e}_1 = \mathrm{b}^{ROM}$$

- This defines the ROM propagator operator $\mathcal{P}_q^{ROM} = \mathrm{R}^{-T}\mathrm{S}\mathrm{R}^{-1}$
- $\mathcal{P}_q^{ROM}$ is a projection of the propagator $\mathcal{P}_q$ onto the orthogonalized snapshots $\mathbf{V}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathrm{R}^{-1}$

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- $\mathcal{P}_q^{ROM}$ is a projection of the propagator $\mathcal{P}_q$ onto the orthogonalized snapshots $\mathbf{V}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathrm{R}^{-1}$
  - Elements of $\mathbf{U}$ are strongly dependent on q
  - Elements of $\mathbf{V}$ are localized and weakly dependent on q
  - $\Rightarrow$ Known background ($q = 0$) snapshots $\mathbf{V}_0(\mathbf{x}) \approx \mathbf{V}(\mathbf{x})$ provide embedding of ROM into physical space

Reduced order model propagator operator II

$$\mathrm{u}_{j+1}^{ROM} = 2\mathcal{P}_q^{ROM}\mathrm{u}_j^{ROM} - \mathrm{u}_{j-1}^{ROM} \qquad \mathrm{u}_0^{ROM} = \mathrm{R}\mathrm{e}_1 = \mathrm{b}^{ROM}$$

- Defines data $\mathbf{D}_j^{ROM} = (\mathrm{b}^{ROM})^T \mathcal{T}_j(\mathcal{P}_q^{ROM})\mathrm{b}^{ROM}$
- Interpretation as discrete-time, discrete-space WEQ

$$\frac{\mathrm{u}_{j+1}^{ROM} - 2\mathrm{u}_j^{ROM} - \mathrm{u}_{j-1}^{ROM}}{\tau^2} = -\frac{2}{\tau^2}(\mathrm{I} - \mathcal{P}_q^{ROM})\mathrm{u}_j^{ROM} = -\mathcal{L}_q^{ROM}(\mathcal{L}_q^{ROM})^T\mathrm{u}_j^{ROM}$$

Reduced order model propagator operator II

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- Compare to to continuous-time, continuous-space WEQ

$$\partial_t^2 \mathbf{u}(t,\mathbf{x}) = -L_q L_q^T \mathbf{u}(t,\mathbf{x})$$

- $\mathcal{L}_q^{ROM}$ can be interpreted as a discretization of L_q absorbing $\mathcal{O}(\tau^2)$ error from time discretion

Interpolation Proof, $\mathbf{D}_j^{ROM} = \mathbf{D}_j \quad j \leq 2n-1$

$j \leq n-1$

- For $j \leq n-1$ the true field is in the span of $\mathbf{U}(x)$
- The Galerkin condition is unique and $\mathrm{g}_j = \mathrm{e}_{j+1}$ (block identity matrix)

$j \leq 2n-2$

- The field is a (Chebyshev) polynomial in $\mathcal{P}_q$ and $\mathcal{P}_q = \mathcal{P}_q^T$
  $\mathbf{u}_j = \mathcal{T}_j(\mathcal{P}_q)\mathbf{b}$
- For $l = 1, \ldots, n-1$ we use the Chebyshev identity
  $\mathcal{T}_{n-1+l} = 2\mathcal{T}_{n-1}\mathcal{T}_l - \mathcal{T}_{|n-1-l|}$

$$
\begin{aligned}
\mathbf{D}_{n-1+l} &= \langle \mathbf{b}, \mathcal{T}_{n-1+l}(\mathcal{P}_q)\mathbf{b} \rangle \\
&= 2\left\langle \mathcal{T}_{n-1}(\mathcal{P}_q)\mathbf{b}, \mathcal{T}_l(\mathcal{P}_q)\mathbf{b} \right\rangle - \left\langle \mathbf{b}, \mathcal{T}_{|n-1-l|}(\mathcal{P}_q)\mathbf{b} \right\rangle \\
&= 2\left\langle \mathbf{u}_{n-1}, \mathbf{u}_l \right\rangle - \left\langle \mathbf{b}, \mathbf{u}_{|n-1-l|} \right\rangle \\
&= 2(\mathrm{u}_{n-1}^{ROM})^T \mathrm{u}_l^{ROM} - \mathrm{b}^{ROM} \mathrm{u}_{|n-1-l|}^{ROM} \\
&= \left\langle \mathbf{b}, \mathcal{T}_{n-1+l}(\mathcal{P}_q^{ROM})\mathbf{b} \right\rangle = \mathbf{D}_{n-1+l}^{ROM}
\end{aligned}
$$

- Similar prove for $j = 2n-1$

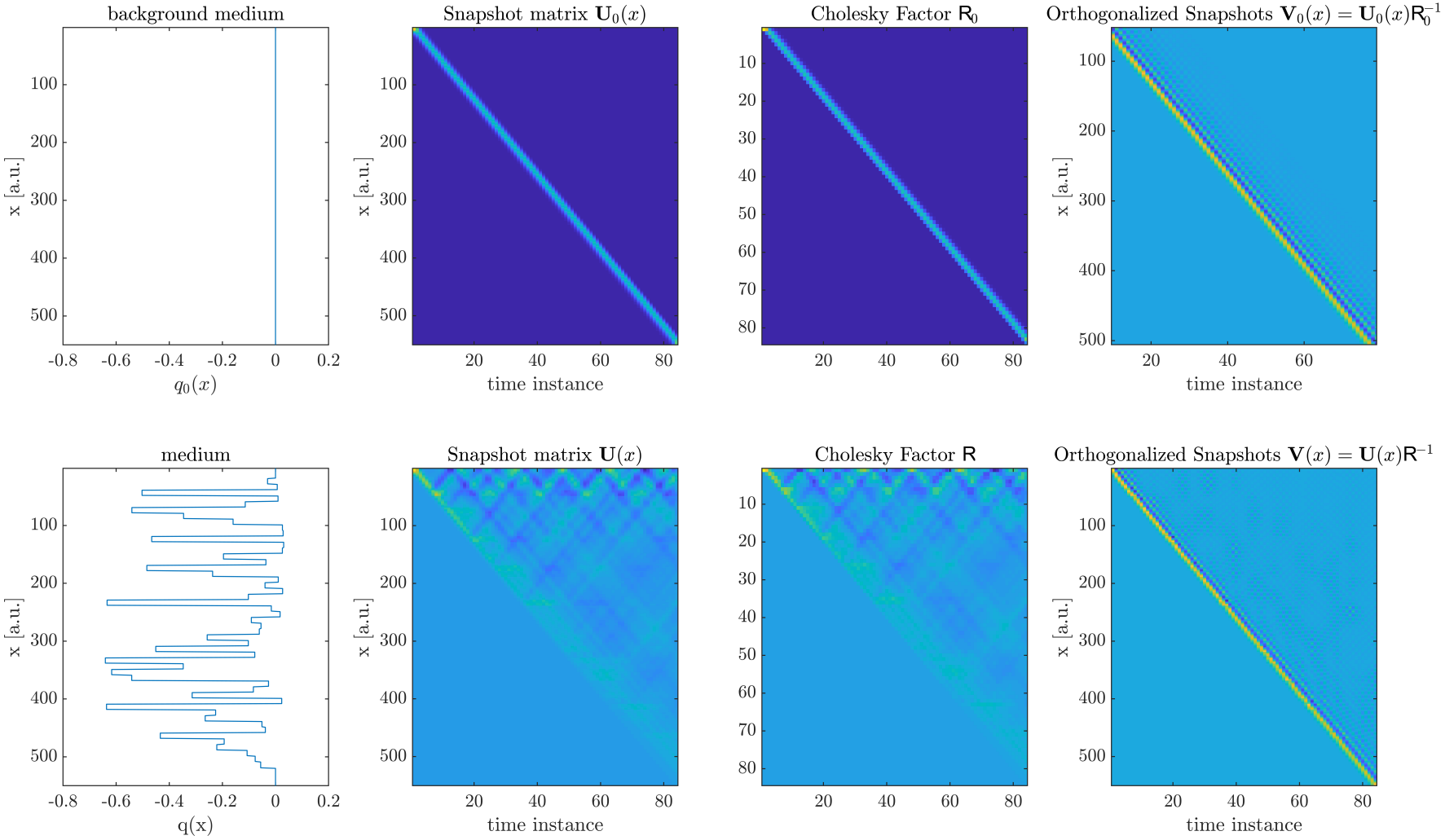
Finding M and S from the Data

- The Mass matrix is defined as $\mathrm{M} = \langle \mathbf{U}, \mathbf{U} \rangle$

$$
\begin{aligned}
(\mathrm{M})_{i+1,j+1} &= \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\
&= \langle \mathcal{T}_i(\mathcal{P}_q)\mathbf{b}, \mathcal{T}_j(\mathcal{P}_q)\mathbf{b} \rangle = \frac{1}{2} \langle \mathbf{b}, 2\mathcal{T}_i(\mathcal{P}_q)\mathcal{T}_j(\mathcal{P}_q)\mathbf{b} \rangle \\
&= \frac{1}{2} \left\langle \mathbf{b}, [\mathcal{T}_{i+j}(\mathcal{P}_q) + \mathcal{T}_{|i-j|}(\mathcal{P}_q)]\mathbf{b} \right\rangle \\
&= \frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|})
\end{aligned}
$$

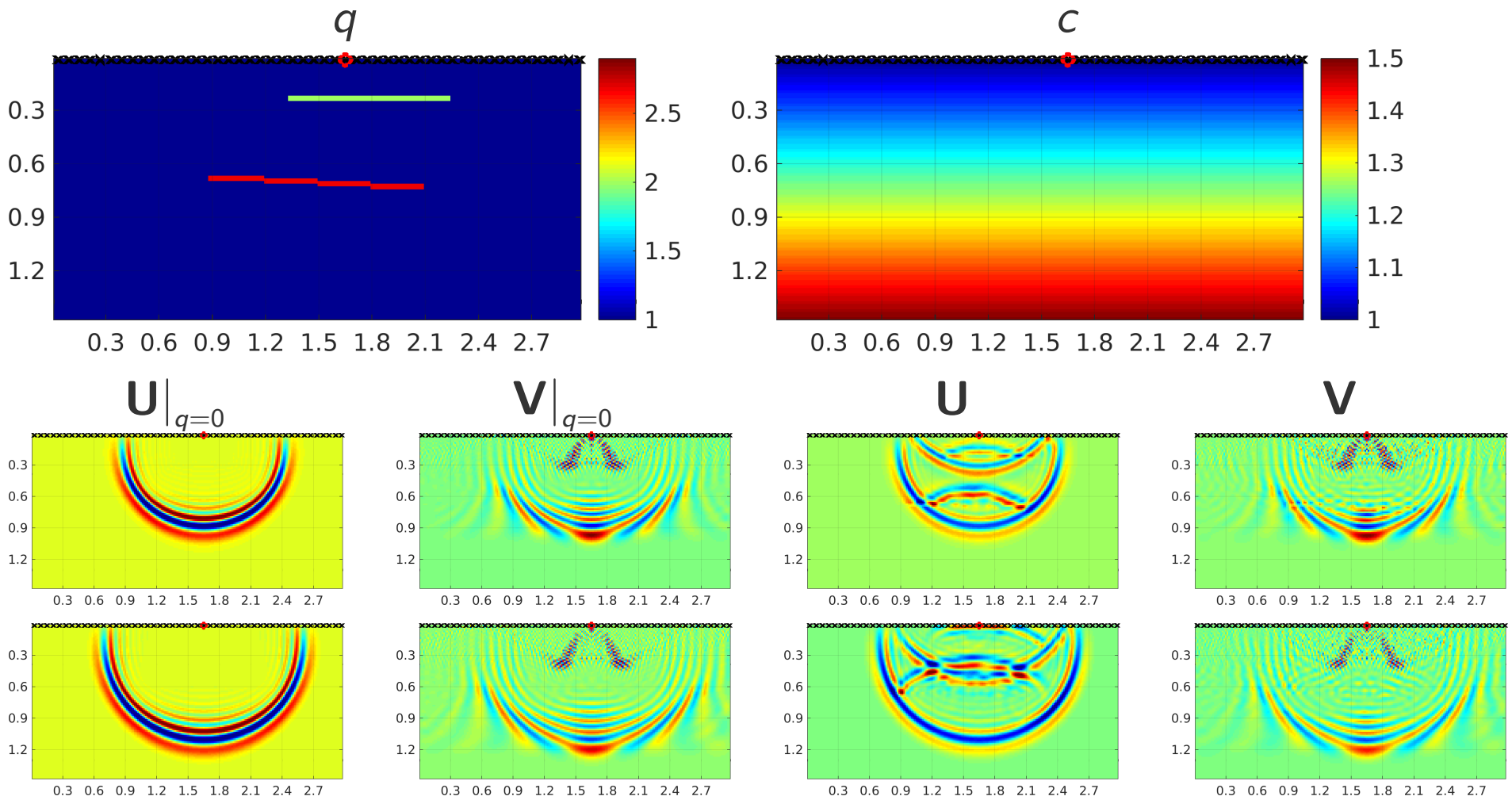
- Similar result can be obtained for S
- With noise M and S need to be regularized such that
  1. M is positive definite
  2. The pencil (M, S) has a maximum eigenvalue ≤ 1 (Guarantees that $\mathcal{P}^{ROM}$ is contractive operator)
- Basically Löwner inner products in the time domain
- Next: $\mathbf{V}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathrm{R}^{-1}$ independence of q

Orthogonalized Snapshot matrix - 1D



- Orthogonalized Snapshots approximately independent of medium

Orthogonalized Snapshot matrix - 2D



Array with $m = 50$ sensors $\times$

Snapshots plotted for a single source $\circ$

Summary ROM

- From the data we can obtain a ROM that explains the data

$$\frac{\mathrm{u}_{j+1}^{ROM} - 2\mathrm{u}_{j}^{ROM} - \mathrm{u}_{j-1}^{ROM}}{\tau^2} = -\mathcal{L}_q^{ROM}(\mathcal{L}_q^{ROM})^T \mathrm{u}_j^{ROM}$$

- Size of the ROM is dictated by the data
- The ROM lives in a basis of orthogonalized snapshots $\mathbf{U}\mathrm{R}^{-1}$ (localization)
- $\mathcal{L}_q^{ROM}(\mathcal{L}_q^{ROM})^T$ block tridiagonal
- $\mathcal{L}_q^{ROM}$ is nearly affine in the reflectivity q
- $\mathcal{L}_q^{ROM} - \mathcal{L}_0^{ROM}$ close to linear in q [1]

$$\mathsf{I} - \mathcal{P}_q = \mathsf{I} - \cos\left(\tau\sqrt{\mathcal{L}\mathcal{L}^T}\right) \approx -\frac{1}{2}\tau^2\mathcal{L}\mathcal{L}^T + \mathcal{O}(\frac{\tau^4}{4!})$$

[1] Liliana Borcea, Vladimir Druskin, Alexander Mamonov and Mikhail Zaslavsky, *Untangling the nonlinearity in inverse scattering with data-driven reduced order models*, Inverse Problems, Volume 34, Number 6

Inversion Method

- Minimize ROM mismatch rather than data mismatch

$$\mathcal{O}^{LS}(q^s) = ||\mathcal{L}_q^{ROM} - \mathcal{L}_{q_s}^{ROM}||_F + \textit{regularization}$$

- We assume a kinematic model c_0 and an initial guess $q_0 = 0$ to obtain $\mathcal{L}_0^{ROM}$
- Parametrize the reflectivity as

$$q^s(x) = \sum_{j=1}^{N^s} q_j^s \phi_j(x)$$

- Use (approximate) affine relationship (approximate as $\mathbf{V}_0 \approx \mathbf{V}$)

$$\mathcal{L}_{q^s}^{ROM} \approx \mathcal{L}_0^{ROM} + \sum_{j=1}^{N^s} q_j^s (\mathcal{L}_{\phi_j}^{ROM} - \mathcal{L}_0^{ROM})$$

- The coefficients q_j follow from a least-squares problem
- Approach can be iterated
- How to choose $\phi_j(x)$? $\Rightarrow$ Based on Resolution

Choosing $\phi_j(x)$

- The resolution of imaging method depends on location in Ω
- A point like perturbation $\delta_j(\mathbf{x})$ at the point $\mathbf{x}_j$ causes a local perturbation of the diff.op. $\Delta L(\delta_j)$ such that

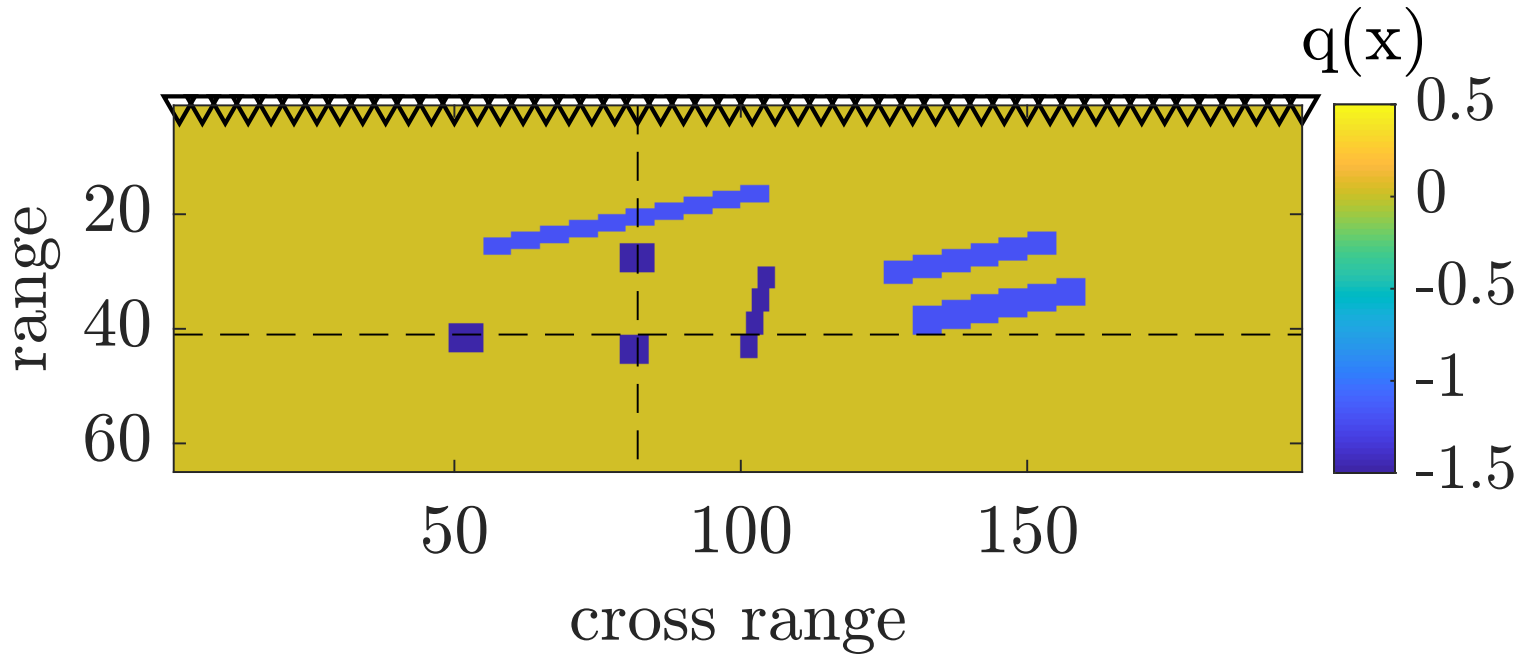
$$\Delta L(\delta_j)\Delta L(\delta_j)^T\varphi(\mathbf{x}) = \frac{c^2(\mathbf{x})}{4}|\nabla\delta_j(\mathbf{x})|^2\varphi(\mathbf{x})$$

- Idea: Lift the perturbation $\Delta\mathcal{L}_{\delta_j}^{ROM}$ that δ_j causes in the ROM into the physical space using the orthogonalized basis function of the background $\mathbf{V}_0(\mathbf{x})$

$$\Psi_j(\mathbf{x}) = \left[\mathbf{V}_0(\mathbf{x})\Delta\mathcal{L}_{\delta_j}^{ROM}(\Delta\mathcal{L}_{\delta_j}^{ROM})^T\mathbf{V}_0^T(\mathbf{x})\right]$$

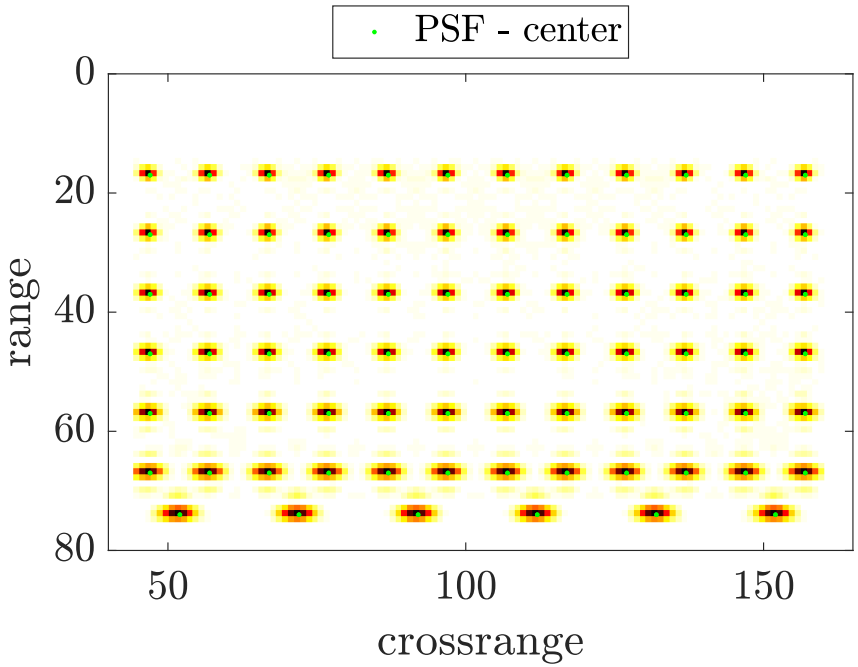
- We do a partition of unity using these point spread function
- If we need to regularize our ROM due to noisy data, it directly impacts this resolution

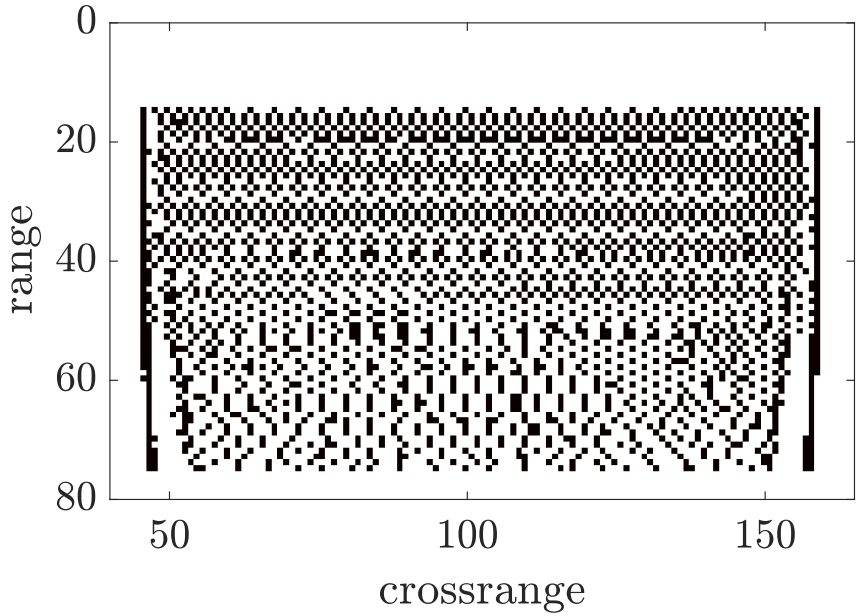
Numerical experiments



- 50 source & receiver pairs
- Derivative of Gaussian pulse with $\lambda_{peak} = 9$ dimensionless units.
- 5% cut-off at $\lambda_{cut} = 4.5$ units
- 110 timesteps
- Constant background velocity

Resolution Analysis - Defining search space



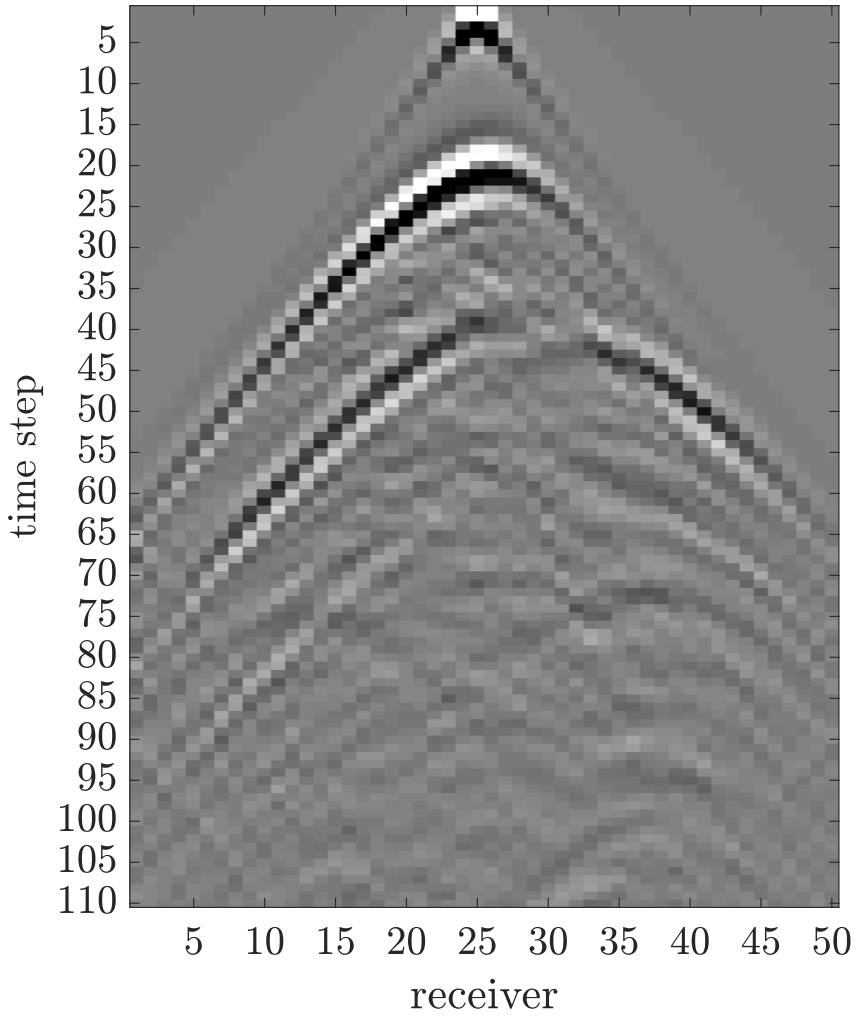


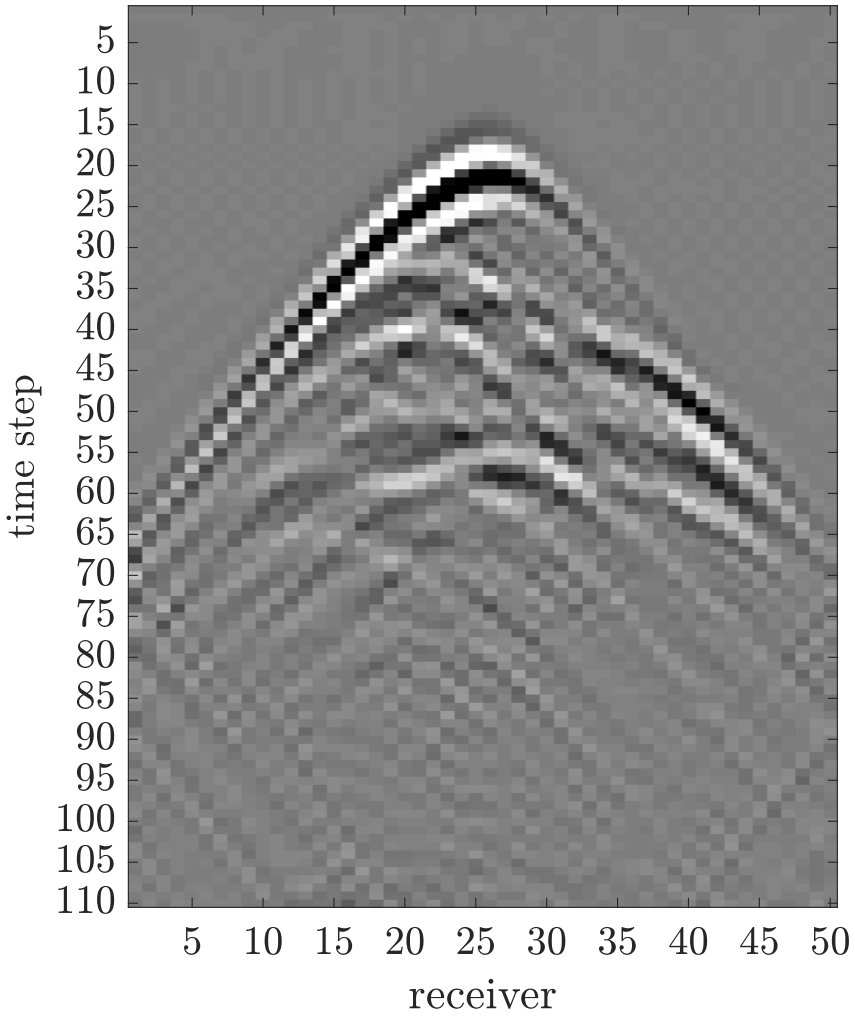
- Examples of point spread functions Ψ_j
- Cross range resolution decreases away from array.

- Centers of basis functions ϕ_j used after a partition of unity of the PSFs

$$\min \|\alpha\|_1, \quad \text{such that}$$
$$\left|1 - \sum_j \alpha_j \Psi_j\right| \leq \text{tol},$$

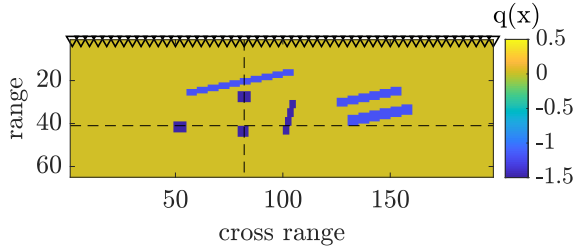
Data

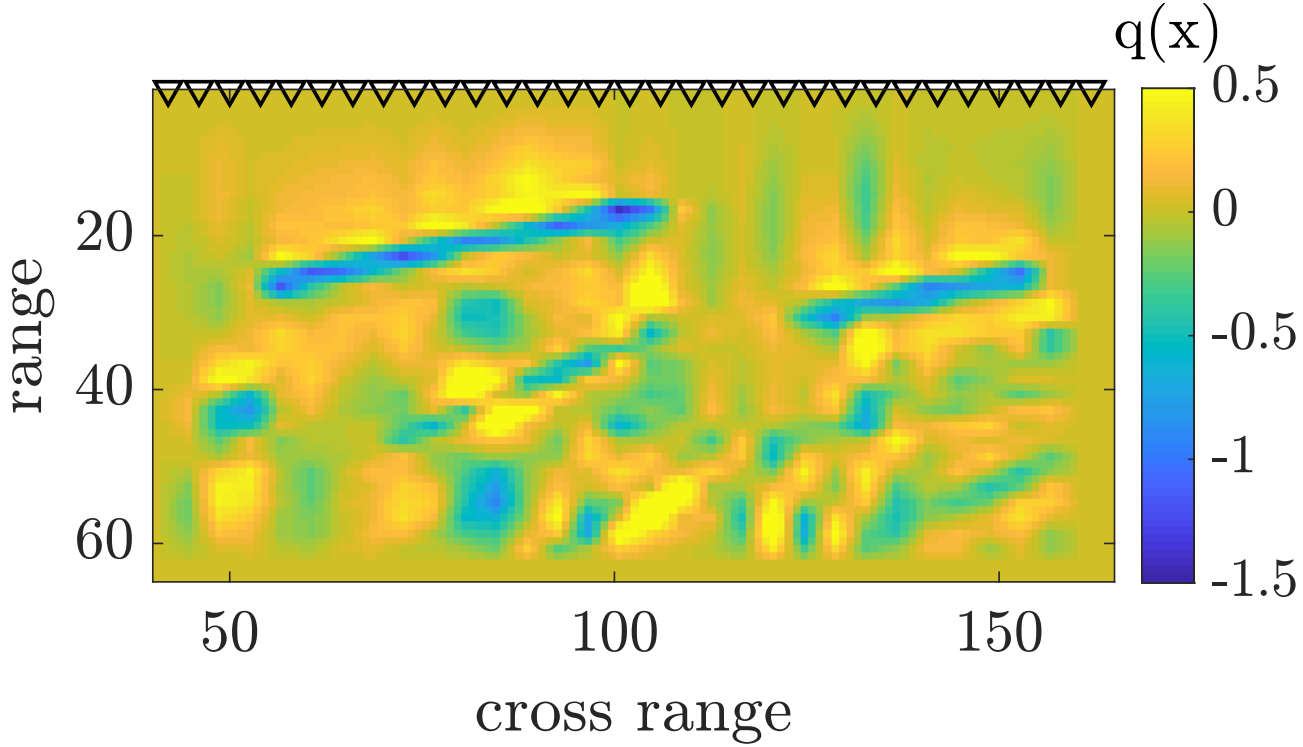




- Data collected by all receivers after firing 25th source.
- Data if the map $q \mapsto \mathbf{D}(t)$ were linear

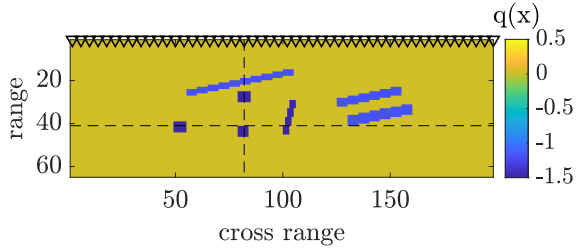
Classical Least squares Data-misfit

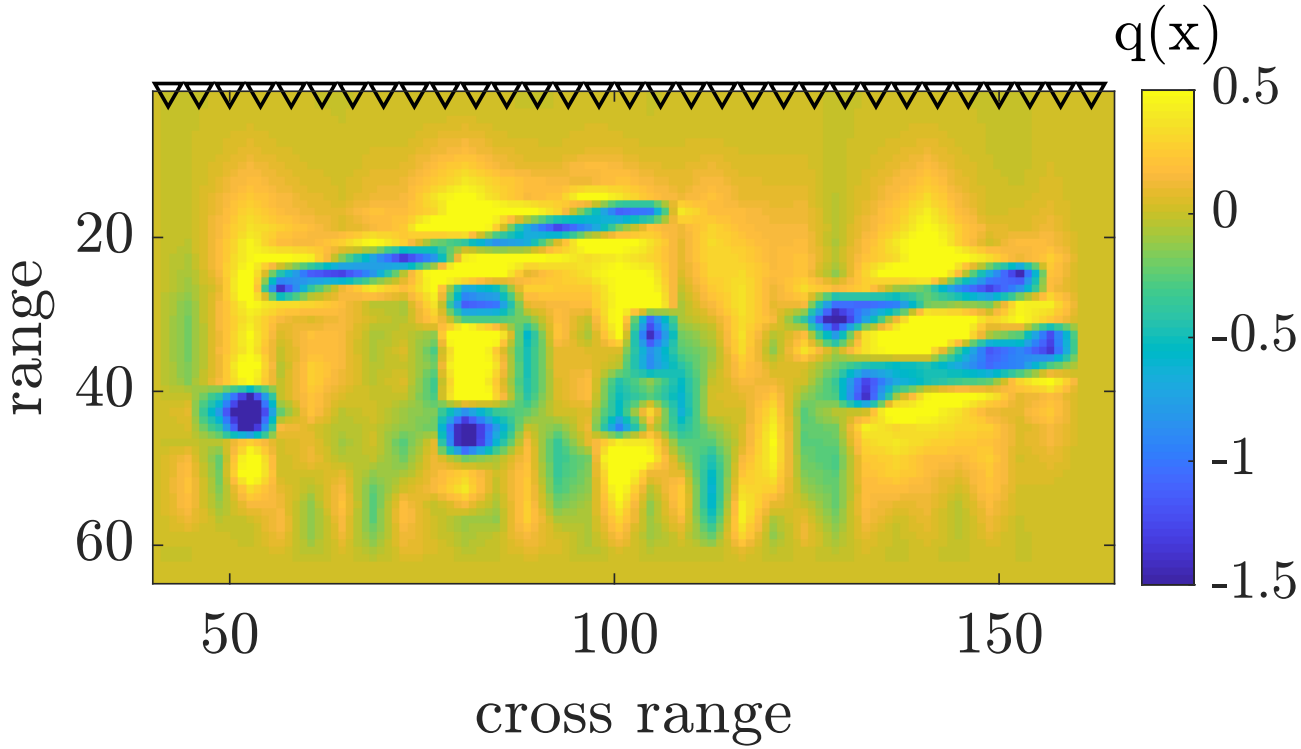




- Data least squares $||\mathbf{D}_q^{meas} - \mathbf{D}_{q_s}^{model}||_F$
- (truncated SVD for regularization)

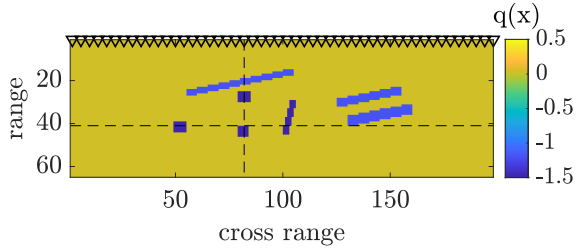
Least squares Born Data

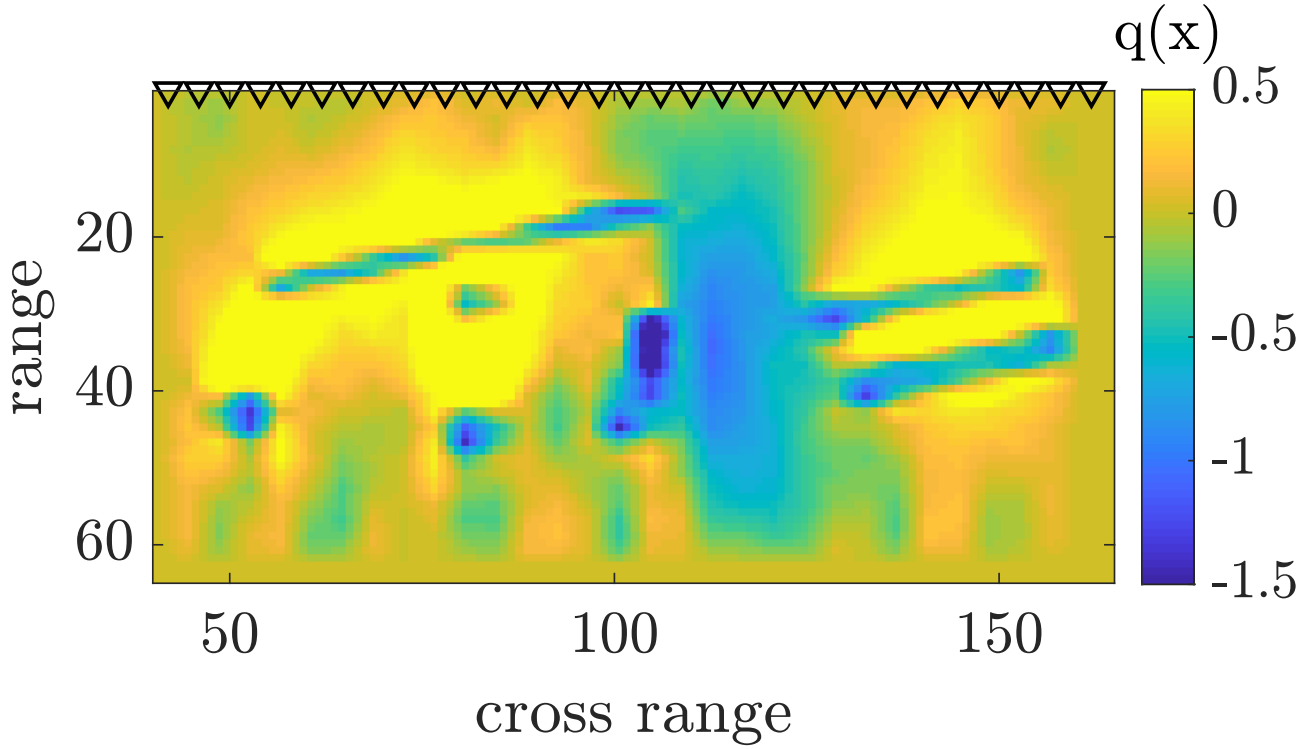




- Data least squares if the map $q \mapsto \mathbf{D}$ were linear

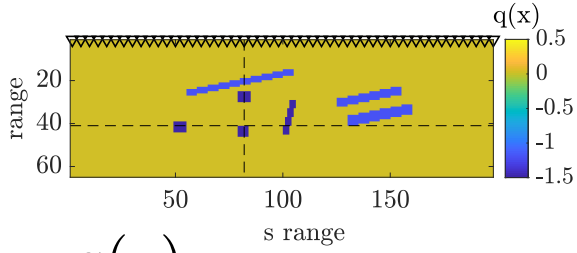
Iterative imaging with ROM it:1

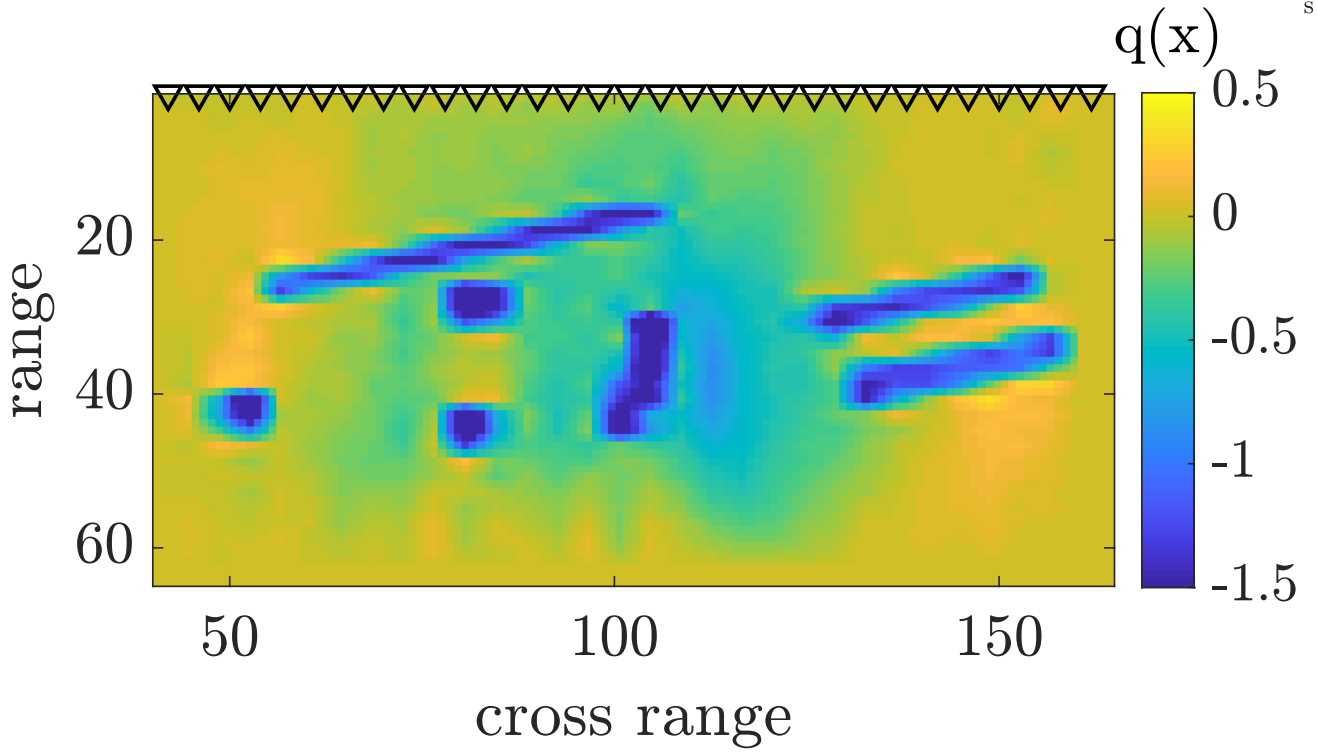




- One iterate of nonlinear LS for $||\mathcal{L}_q^{ROM} - \mathcal{L}_{q_s}^{ROM}||_F$
- Qualitative agreement

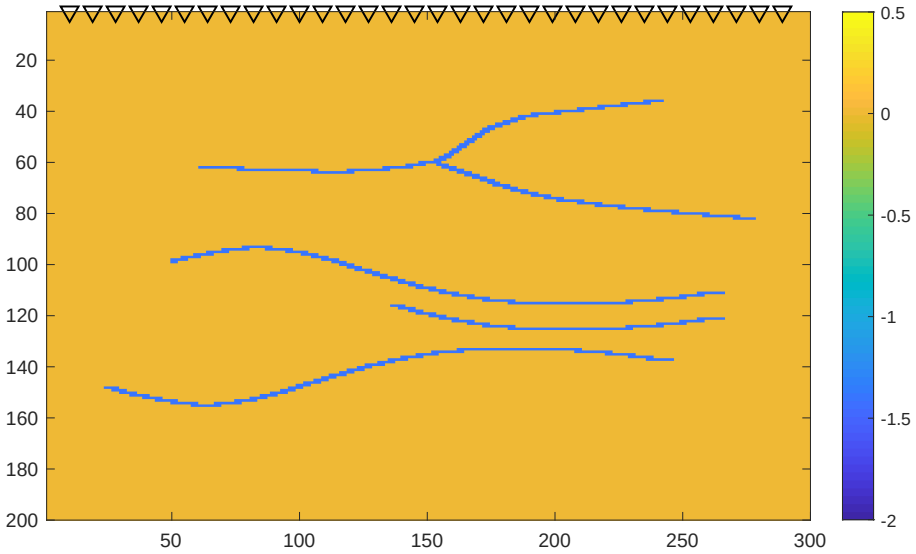
Iterative imaging with ROM it:5

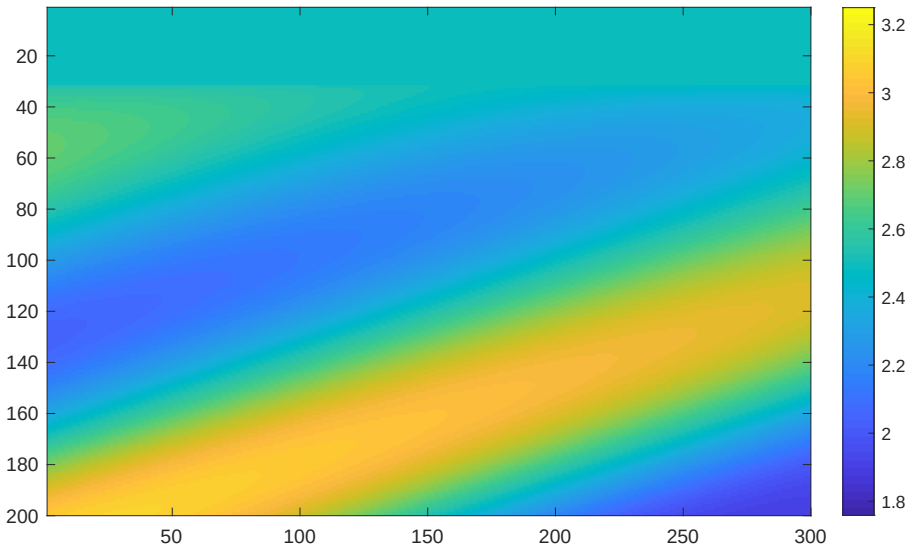




- Multiple reflections hold information, i.e. better image than with linear data
- Quantitative agreement

Cracks Example - added noise

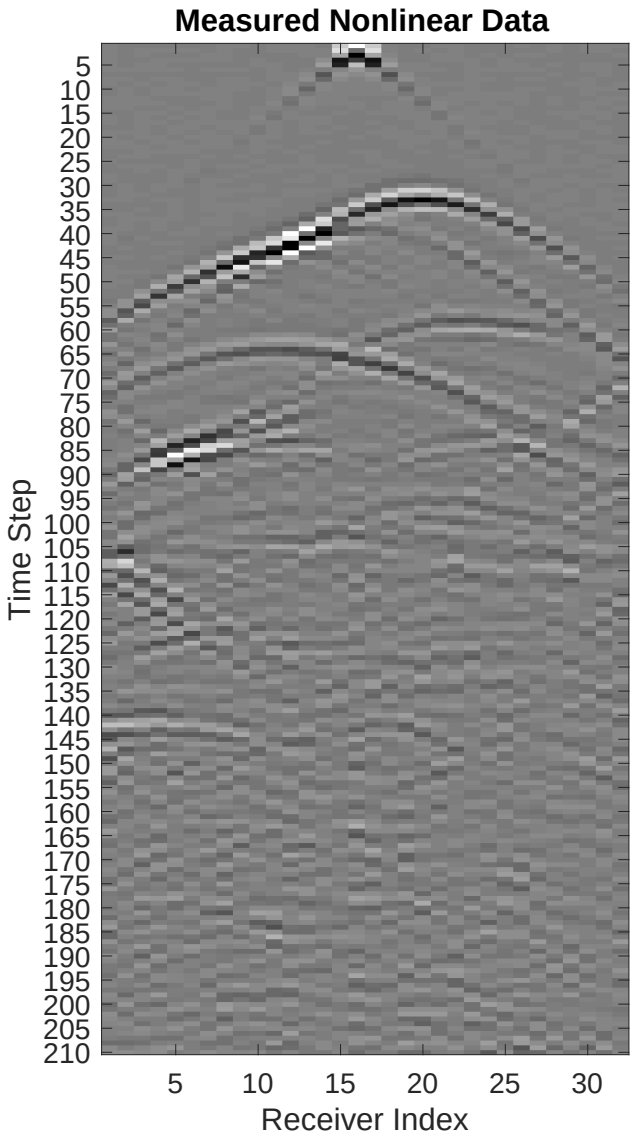




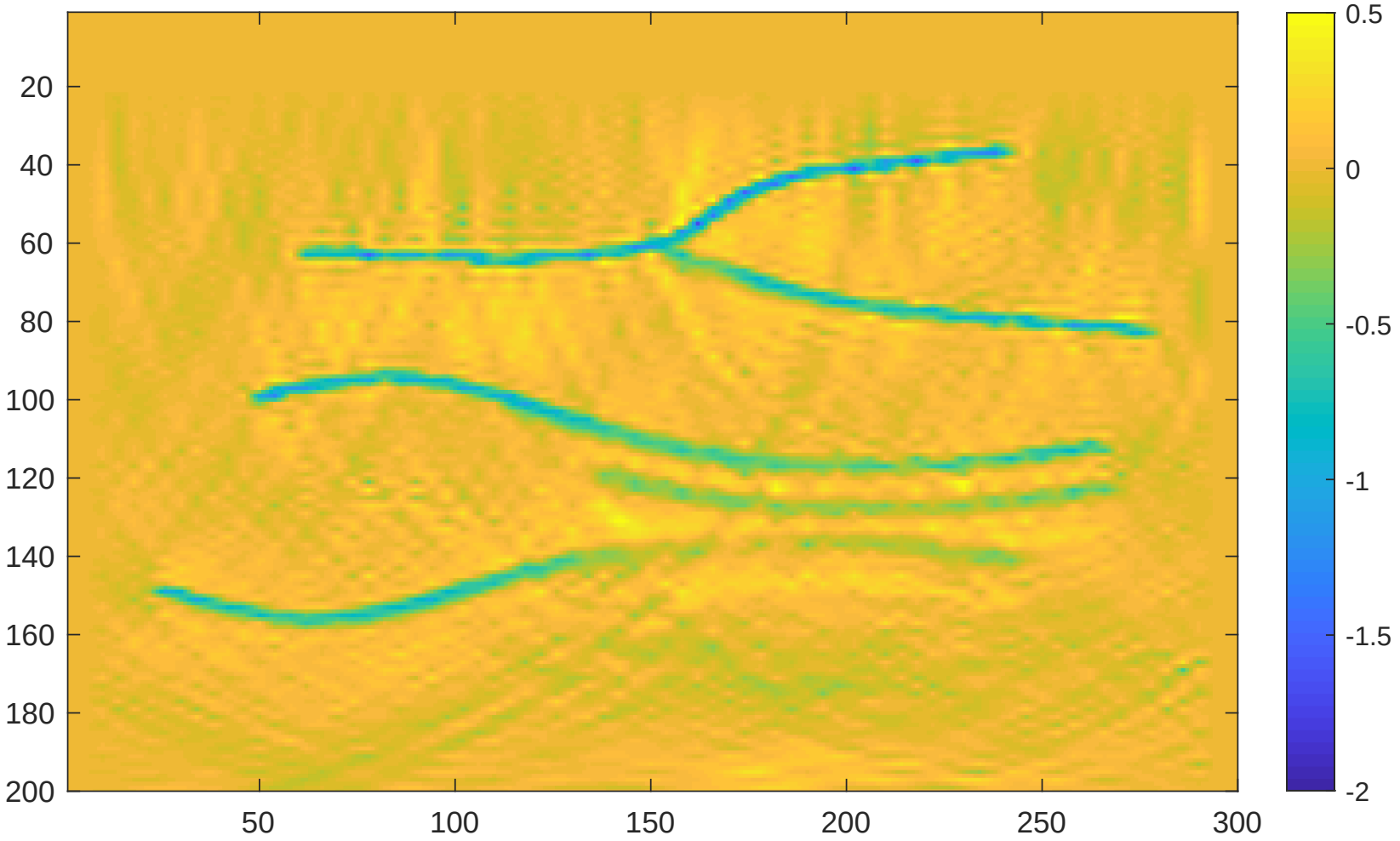
- Example with 5% white measurements noise
- Gaussian pulse with $\lambda_{peak} = 10 - 15$ units
- Width of cracks 2 units

Background speed

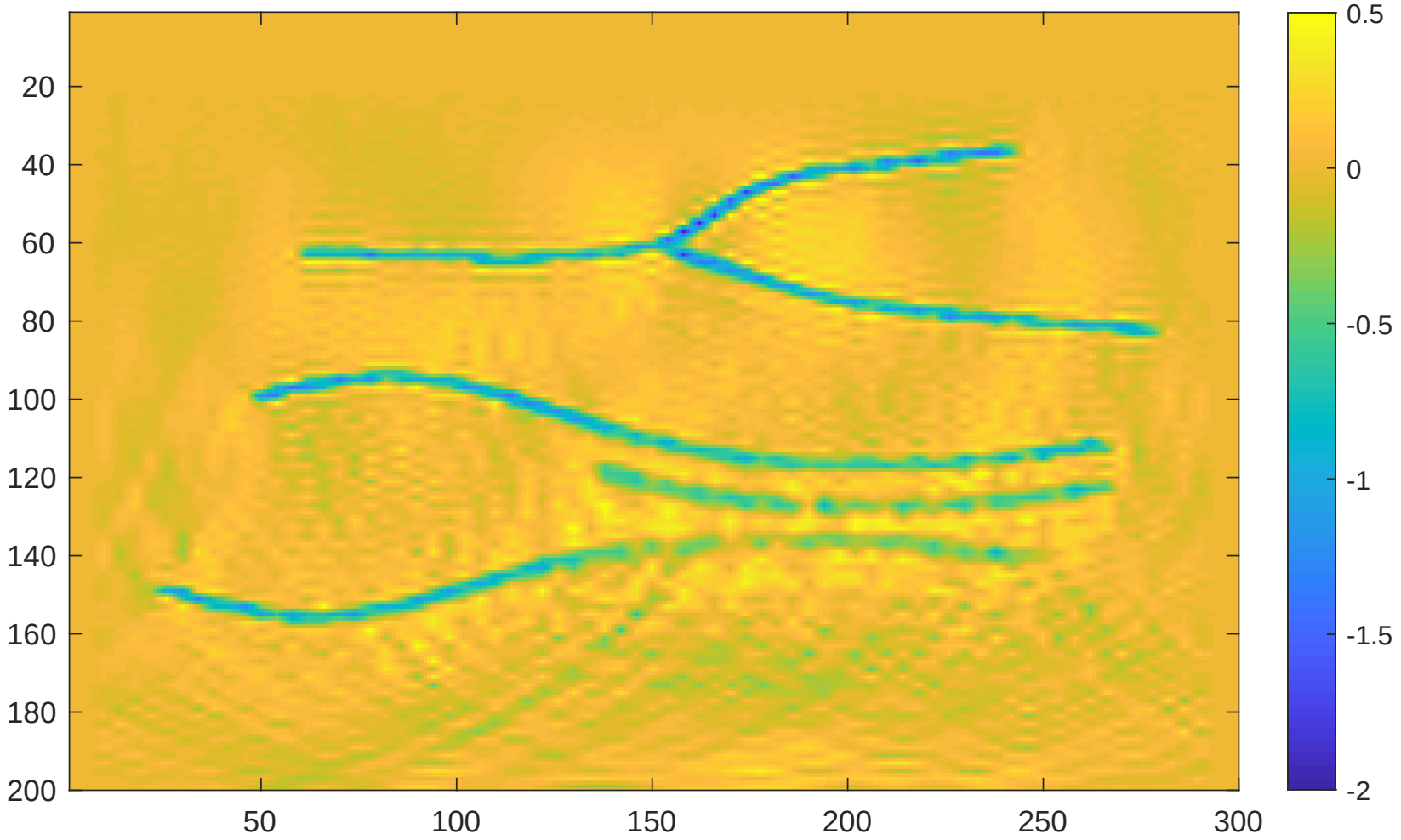
Cracks Example Measurements



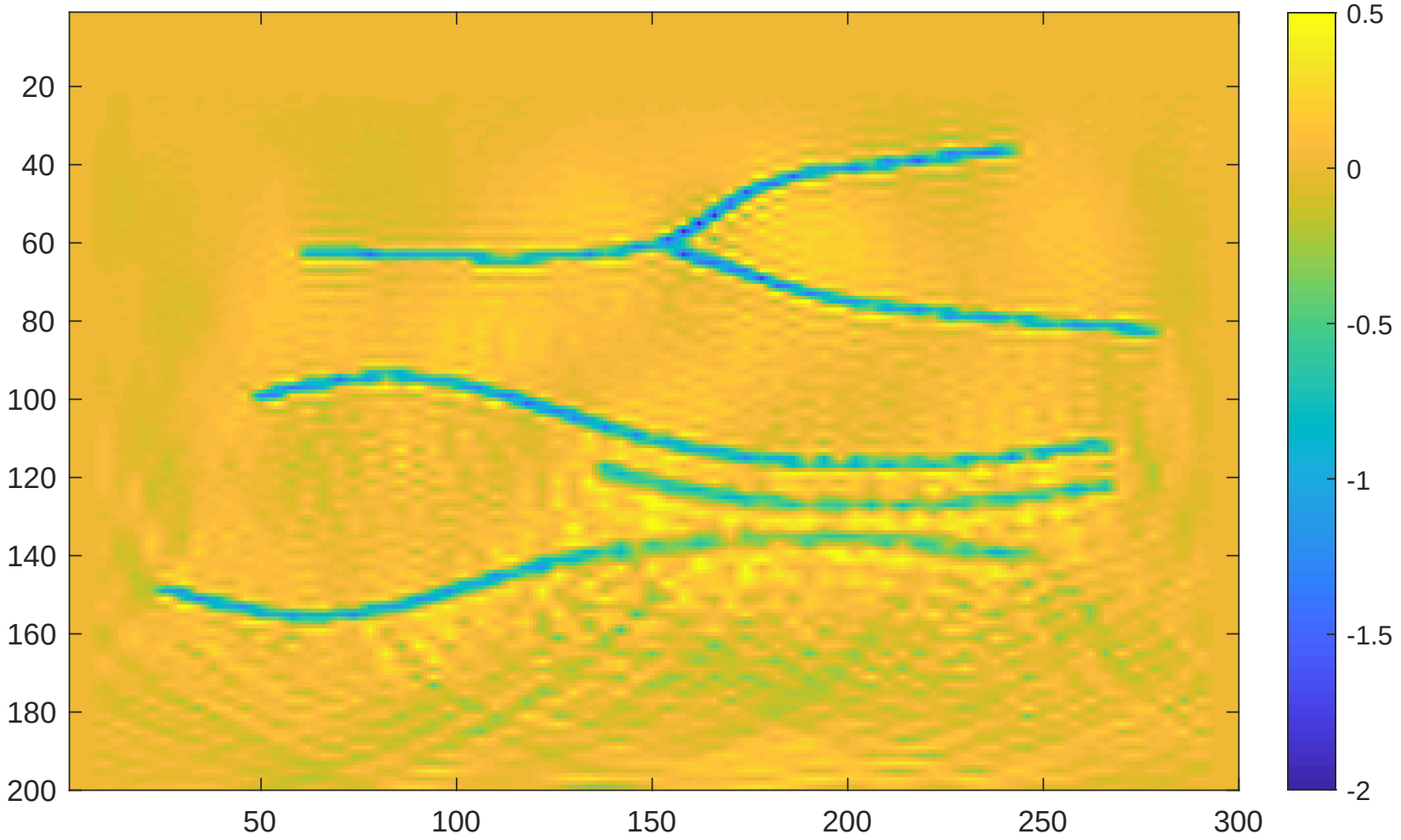
Cracks Iteration 1



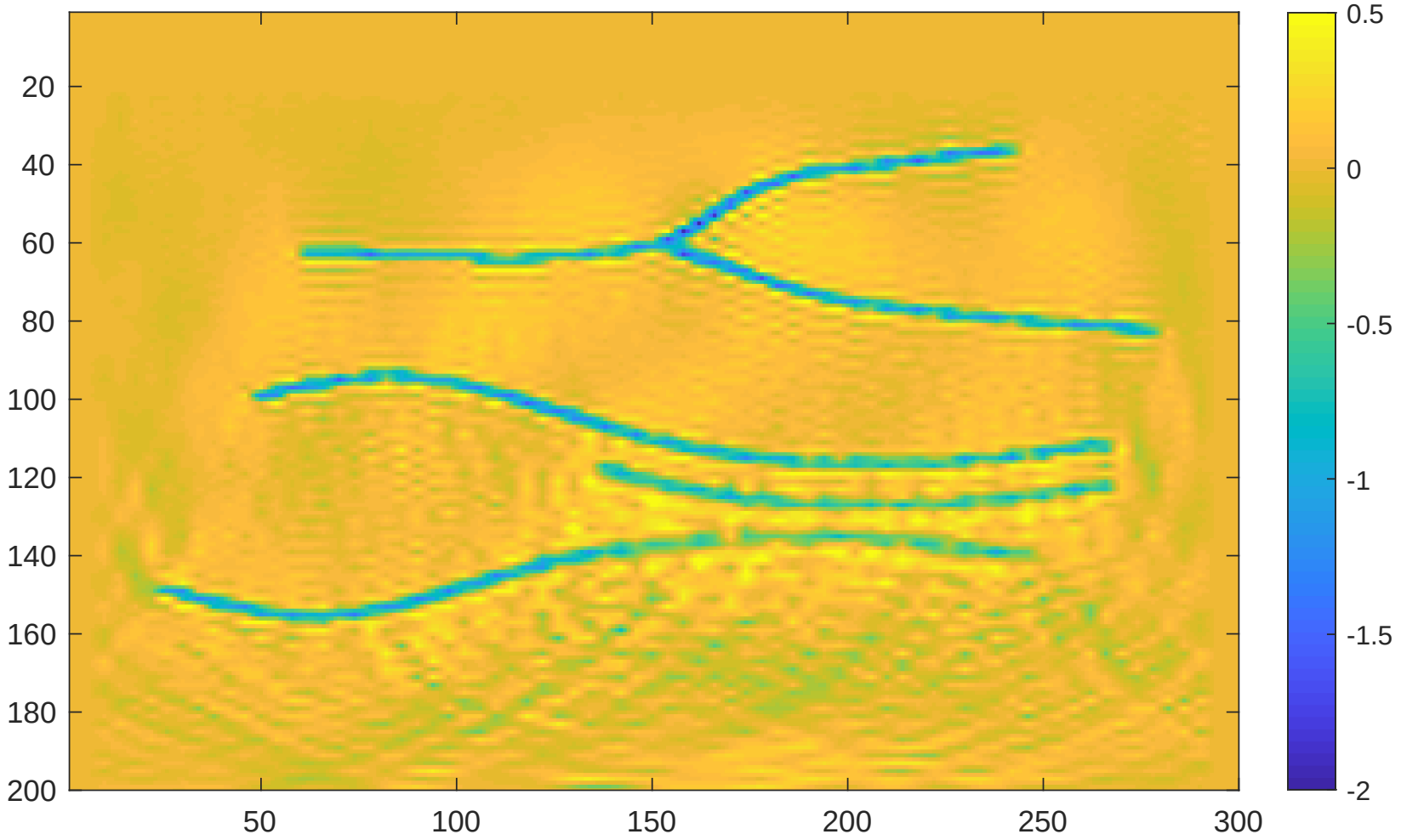
Cracks Iteration 2



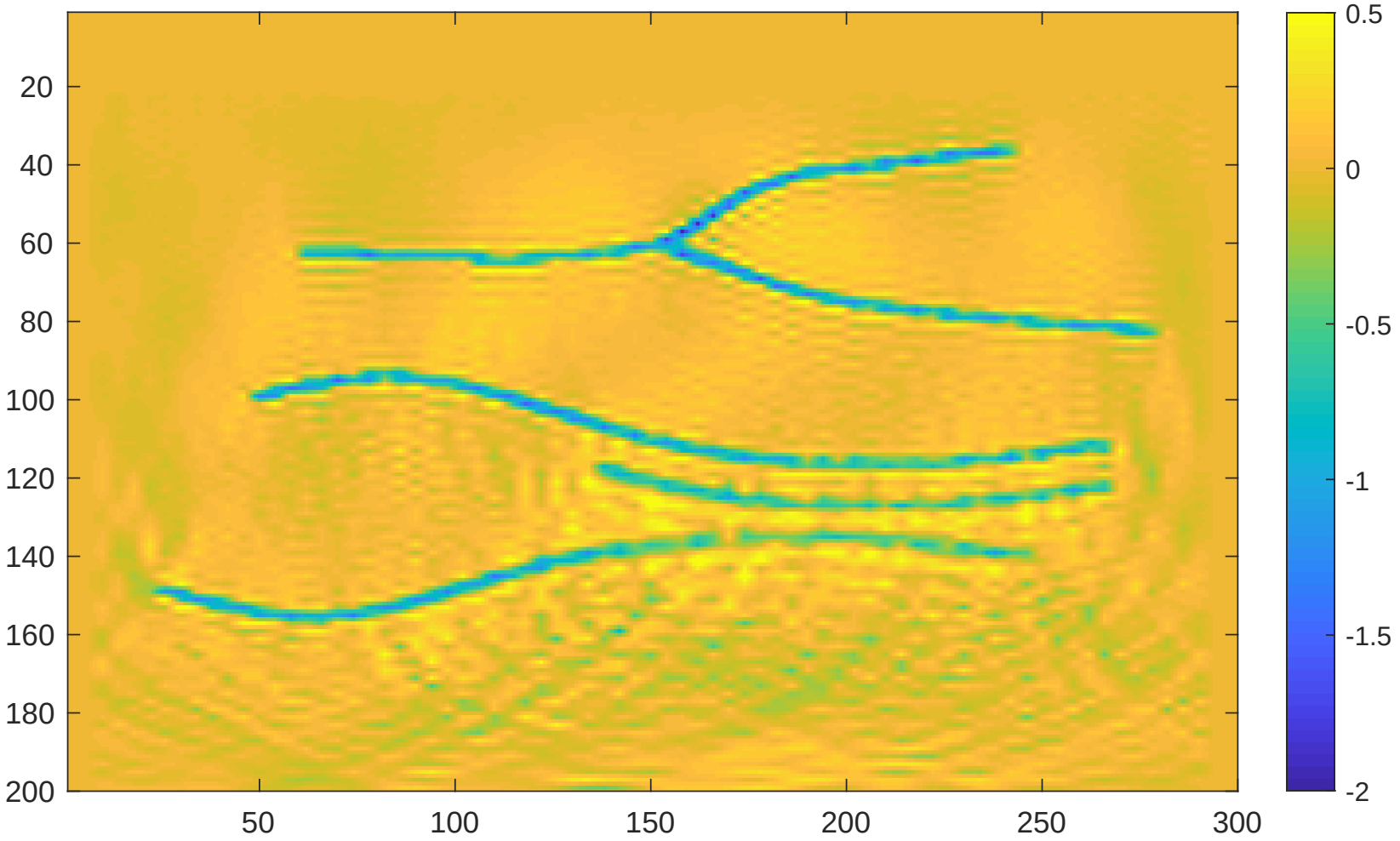
Cracks Iteration 3



Cracks Iteration 4

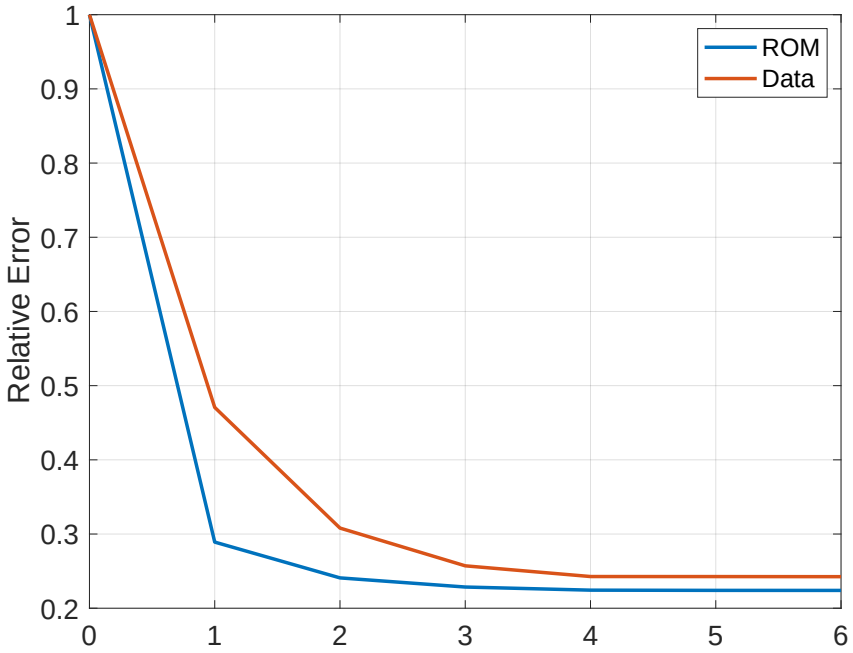


Cracks Iteration 5



Reconstruction metrics

- All cracks are recovered and well seperated
- Less sensitivity far away from array
- Convergence in 3 iterations



Conclusions

1. The objective function

$$||\mathcal{L}_q^{ROM} - \mathcal{L}_{q^s}||_F$$

   is close to linear in the unknown reflectivity q. Few iterations are needed to recover q from this functional
2. Intrinsic regularization via the ROM formulation
3. ROM: Pure linear algebraic method on the level of the data

Reduced Order Model Approach to Inverse Scattering, L. Borcea, V. Druskin, A.V. Mamonov, M. Zaslavsky, J. Zimmerling, to appear in SIAM Journal on Imaging Sciences, 2020. Preprint: arXiv:1910.13014 [math.NA]

Reduced Order Model Approach to Inverse Scattering

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22 April 2020, ICERM & Jörn's Couch